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## STRING DUALITY AND MODULAR FORMS

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### ABSTRACT

Tests of duality between heterotic strings on  $K3 \times T^2$  (restricted on certain Narain moduli subspaces) and type IIA strings on  $K3$ -fibered Calabi-Yau threefolds are attempted in the weak coupling regime on the heterotic side by identifying pertinent modular forms related to the computations of string threshold corrections. Concretely we discuss in parallel the three cases associated with Calabi-Yau manifolds  $(A) : X(6, 2, 2, 1, 1)_2^{-252}$ ,  $(B) : X(12, 8, 2, 1, 1)_3^{-480}$  and  $(C) : X(10, 3, 3, 2, 2)_4^{-132}$  on the type IIA side.

In the past year it has become harder and harder to deny that many string theories allow dual descriptions. Through this still ongoing development, invaluable information about non-perturbative facets of string theory has been accumulated. String compactifications that exhibit  $N = 2$  supersymmetry in four dimensions offer particularly interesting class of examples in such string duality phenomena [1–12]; heterotic strings compactified on  $K3 \times T^2$  may have dual type IIA theories on Calabi-Yau manifolds that admit structures of  $K3$ -fibrations [3]. An extensive list of  $K3$ -fibered Calabi-Yau manifolds has recently been given in [13].

In this article we shall be concerned with three (possible)  $N = 2$  heterotic-type IIA pairs. On the type IIA side these correspond to the following  $K3$ -fibered Calabi-Yau threefolds\*:

$$\begin{aligned} (A): \quad & X(6, 2, 2, 1, 1)_2^{-252} \\ (B): \quad & X(12, 8, 2, 1, 1)_3^{-480} \\ (C): \quad & X(10, 3, 3, 2, 2)_4^{-132} \end{aligned}$$

The first two cases are most familiar and candidate heterotic duals were discovered in the pioneering work of Kachru and Vafa [1]. In [7] a certain heterotic string vacuum was considered in connection with (B). This vacuum is not the one considered in [1] for (B), thus it is not precisely the dual of type IIA string on (B). However, in refs. [7, 10], it was convincingly pointed out that perturbative calculations restricted to a particular Narain moduli *subspace* match up well with the type IIA calculations on (B). For the third model, although we have not yet found a precise heterotic dual in the strict sense of Kachru and Vafa, we should like to pursue a similar story as in [7, 10]. In the following we will describe a possible heterotic vacuum which seems to be related to the type IIA string on (C) when restricted to a certain moduli subspace. (Thus, morally speaking, this description should be understood as serving a motivation to write down the expression given later.) An  $\mathcal{E}_8 \times \mathcal{E}_8$  heterotic string compactified on  $K3 \times T^2$  with standard embedding has generically gauge symmetry of  $\mathcal{E}_8 \times \mathcal{E}_7 \times U(1)^4$  if we include graviphoton. In total we have  $248 + 133 + 4 = 385$  vector states. The massless spectrum also contains 625 hypermultiplets – 10 hypermultiplets belonging to **56** of  $\mathcal{E}_7$ , the 20  $K3$  moduli hypermultiplets and the 45 gauge bundle moduli hypermultiplets. To give some idea of the relevance of (C), recall that the gauge symmetry  $\mathcal{E}_7$  is attained through enhancement of symmetry from the maximal subgroup  $SO(12) \times SU(2)$  where  $SO(12)$  is realized by twelve free left gauge fermions on the world-sheet (if we adopt the fermionic formulation) and  $SU(2)$  stems from the  $N = 4$  superconformal algebra of the  $K3$  sigma model. We note that  $\mathcal{E}_7$  irreps **133** and **56** are decomposed under  $SO(12) \times SU(2)$  as **133**  $\rightarrow$  (**32<sub>c</sub>**, **2**) + (**1**, **3**) + (**66**, **1**) and **56**  $\rightarrow$  (**32<sub>s</sub>**, **1**) + (**12**, **2**). Now suppose we go to the Coulomb branch of this  $SU(2)$  but

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\* Here  $X(w_0, \dots, w_4)_{h^{1,1}}^\chi$  denotes the Calabi-Yau manifold with the Hodge number  $h^{1,1}$  and the Euler characteristic  $\chi$  obtained by a hypersurface of degree  $\sum_i w_i$  in  $\mathbf{WP}(w_0, \dots, w_4)$ .

retain the other non-abelian gauge symmetries  $\mathcal{E}_8 \times SO(12)$ , thus considering a particular subspace of the full Narain moduli space. In this moduli subspace the number of vector fields is  $385 - 32 \cdot 2 - (3 - 1) = 319$  and the number of hypermultiplets is  $625 - 10 \cdot 12 \cdot 2 = 385$ . Hence twice their difference is  $2 \cdot (319 - 385) = -132$ . On the other hand, the number of the abelian vector fields whose scalar components consist of the moduli fields of this Narain moduli subspace and the dilaton, is 4. Thus this (restricted) heterotic vacuum is expected to be of relevance to the type IIA string on a  $K3$ -fibered Calabi-Yau threefold with  $h^{1,1} = 4$  and  $\chi = -132$ .

If we go to the Coulomb branch of  $\mathcal{E}_8$  on the heterotic side for the above three cases (A), (B) and (C), we will get theories with the number of abelian vector multiplets (parametrizing the pertinent moduli subspaces) increased by eight. For these theories possible dual type IIA theories can be identified without difficulty. They correspond to the following  $K3$ -fibered Calabi-Yau manifolds<sup>†</sup>:

$$\begin{aligned} (A'): & \quad X(30, 20, 8, 1, 1)_{10}^{-732} \\ (B'): & \quad X(42, 28, 12, 1, 1)_{11}^{-960} \\ (C'): & \quad X(30, 16, 12, 1, 1)_{12}^{-612} \end{aligned}$$

For instance, in the case (C') the counting on the heterotic side goes as follows: the number of relevant abelian vector multiplets is  $4 + 8 = 12$  and the difference between the number of vector fields and that of the hypermultiplets is  $[385 - (248 - 8) - 32 \cdot 2 - (3 - 1)] - [625 - 10 \cdot 12 \cdot 2] = -306$ , thus the predicted Euler characteristic is  $2 \cdot (-306) = -612$ .

As has been vigorously studied over the years [14], for type IIA string compactified on a (not necessarily  $K3$ -fibered) Calabi-Yau manifold, the powerful techniques of mirror transformations [15] make it possible, if  $h^{1,1}$  is sufficiently small, the non-perturbatively exact computation of the prepotential<sup>‡</sup>

$${}^{\text{II}}\mathcal{F}(t) = \frac{1}{3!} \sum_{i,j,k} \kappa_{ijk} t^i t^j t^k + \frac{1}{(2\pi i)^3} \sum_{d \in \mathbf{S}} N^r(d) \text{Li}_3(\mathbf{e}[d \cdot t]) , \quad (1)$$

as well as the topological one-loop free energy [16]

$${}^{\text{II}}F_1^{\text{top}}(t) = -\frac{2\pi i}{12} \int c_2 \wedge (t \cdot J) + \frac{1}{6} \sum_{d \in \mathbf{S}} N^{r,e}(d) \text{Li}_1(\mathbf{e}[d \cdot t]) , \quad (2)$$

$$N^{r,e}(d) = N^r(d) + 12 \sum_{d' \in \mathbf{S}, d' \leq d} N^e(d') , \quad (3)$$

where  $t = (t^1, \dots, t^{h^{1,1}})$  are the Kähler moduli parameters and  $J = (J_1, \dots, J_{h^{1,1}})$  are the integral generators of the complexified Kähler cone. In the above we have introduced a *partially ordered* set  $(\mathbf{S}, \leq)$  where  $\mathbf{S} = \mathbf{Z}_{\geq 0}^{h^{1,1}} \setminus \{\mathbf{0}\}$  and  $d' \leq d$  ( $d, d' \in \mathbf{S}$ )  $\Leftrightarrow \exists n \in \mathbf{Z}_{>0}$  s.t.  $d =$

<sup>†</sup>These Calabi-Yau manifolds and their associated heterotic duals have already appeared in refs. [6, 7].

<sup>‡</sup> Here the  $\kappa_{ijk}$  are the triple intersection numbers and we have omitted lower order polynomial terms. The polylogarithm  $\text{Li}_k$  is defined by  $\text{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}$ . We will write  $\mathbf{e}[x]$  for  $e^{2\pi i x}$ .

$nd'$ . The integers  $N^r(d)$  and  $N^e(d)$  count the virtual numbers of rational and elliptic world-sheet instantons of multidegree  $d$ . Note that  $N^e(d) = \frac{1}{12} \sum_{d' \leq d} \mu(d', d) [N^{r,e}(d') - N^r(d')]$  where  $\mu(\cdot, \cdot)$  is the Möbius function on  $\mathbf{S}$ .

If the  ${}^{\text{II}}\mathcal{F}$  and  ${}^{\text{II}}F_1^{\text{top}}$  of a given  $K3$ -fibered Calabi-Yau manifold are exactly known and if we have already identified a dual heterotic string theory, it is most desirable to compare these type IIA results with heterotic perturbation theory which is not corrected beyond one-loop. Such comparison has been attempted in refs. [2, 4, 6, 7] with satisfactory results.

In particular, the work of Harvey and Moore [7] has made it clear that a beautiful picture emerges in perturbation theory of heterotic strings on  $K3 \times T^2$ : *the coefficients of modular forms appearing in the calculation of threshold corrections are related to rational and elliptic instanton numbers in the type IIA setting and in some cases are also related to the root multiplicities of generalized Kac-Moody (super) algebras*. Thus, it seems that understanding this trinity of apparently remote mathematical concepts is one of the keys to unravel the mystery of string duality.

In this work we will attempt to pursue this line of thoughts for the cases (A), (B) and (C) in parallel. We will present the explicit formulas of relevant modular forms and relate their coefficients to perturbative heterotic prepotentials and gravitational Wilsonian couplings which are to be compared with  ${}^{\text{II}}\mathcal{F}$  and  ${}^{\text{II}}F_1^{\text{top}}$  in the tests of duality conjectures. Unfortunately since the type IIA calculation of  ${}^{\text{II}}\mathcal{F}$  and  ${}^{\text{II}}F_1^{\text{top}}$  for (C) seems not to be available in the literature (and I have not tried to work it out by myself), the test of string duality for (C) is yet to be completed. We leave this as a future problem. We should also mention that part of our results (especially for (B)) is somewhat repetitive<sup>§</sup>. This is to give a unified treatment for all the cases (A), (B) and (C). We will also briefly touch upon the cases (A'), (B') and (C').

Let us begin by reviewing the perturbative moduli space of heterotic string on  $K3 \times T^2$  [7, 17, 18]. The Narain lattice  $M$  of signature  $(2, r)$  may be decomposed as  $M = H \oplus \Lambda$ . Here  $H$  is the unique even unimodular lattice of signature  $(1, 1)$  generated by  $e_1$  and  $e_2$  whose Gram matrix is given by  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $\Lambda$  is a rational<sup>¶</sup> lattice of signature  $(1, r-1)$  which becomes integral after being suitably scaled. Consider

$$\mathcal{D} = \{[\omega] \in \mathbf{P}(M \otimes \mathbf{C}) \mid \omega^2 = 0, \quad \omega \cdot \bar{\omega} > 0\}, \quad (4)$$

and one of its connected component  $\mathcal{D}^+$ . Then  $\mathcal{D}^+ = \Lambda \otimes \mathbf{R} + iC^+(\Lambda) \simeq O(2, r)/(O(2) \times O(r))$  where  $C^+(\Lambda)$  is one of the components (*future light cone*) of  $C(\Lambda) := \{x \in \Lambda \otimes \mathbf{R} \mid x^2 > 0\}$ . The Narain moduli space is  $\mathcal{D}^+$  divided by the  $T$ -duality group (= the automorphism group of the Narain lattice). The hermitian symmetric space  $\mathcal{D}^+$  is also known as a bounded domain of type IV and typically appears as the domain of the period

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<sup>§</sup>The case (B) has been treated in refs. [7, 10].

<sup>¶</sup>Here “rational” means that the entries of the Gram matrix of a basis are rational.

map of a  $K3$ . (See for instance [19, 20].) In this case, the conditions  $\omega^2 = 0$  and  $\omega \cdot \bar{\omega} > 0$  in (4) are the Riemann-Hodge bilinear relations, the (suitably scaled)  $M$  is the lattice of transcendental 2-cycles and the  $T$ -duality group is the monodromy group of the period map. The appearance of the  $K3$  moduli space may be foreseen in view of heterotic-type IIA duality since the perturbative regime of heterotic string is, in the type IIA picture, the region where the base  $\mathbf{P}^1$  of the  $K3$  fibration blows up and only the fiber  $K3$  becomes relevant [8, 9]. To be more precise, when we interpret  $\mathcal{D}^+$  as the domain of a period map, the relevant  $K3$  is a *mirror* [20] of the fiber  $K3$  of the  $K3$ -fibration in the type IIA setting. In terms of this fiber  $K3$  the (suitably scaled)  $\Lambda$  is the lattice of algebraic cycles, *i.e.* the Picard lattice.

Notice that for  $y \in \mathcal{D}^+$ , the  $\omega$  in (4) is parametrized by

$$\omega(y) = e_1 - \frac{y^2}{2}e_2 + y, \quad (5)$$

since  $(\omega \cdot \bar{\omega})(y) = 2(\text{Im } y)^2 > 0$ . This parametrization can easily be understood on the heterotic side if we recall the classical prepotential is given by  ${}^{\text{het}}\mathcal{F}_{\text{cl}} = \frac{1}{2}Sy^2$  where  $S$  is the dilaton and take the  $T$ -duality manifest basis of the period vector after a suitable symplectic transformation [17]. Then  $\omega$  is the *electric* part of the period vector.

In the perturbative calculations on the heterotic side, say those of threshold corrections, the following (manifestly  $T$ -duality invariant) formulas of the spectrum are important:

$$p_R^2 - p_L^2 = \lambda^2, \quad (6)$$

$$\frac{1}{2}p_R^2 = \frac{|\lambda \cdot \omega|^2}{\omega \cdot \bar{\omega}}, \quad \omega = \omega(y), \quad (7)$$

where  $(p_R, p_L)$  are the right-left momenta of the compactified sector and  $\lambda \in M$  and  $y \in \mathcal{D}^+$ . The second formula gives the mass formula of BPS saturated string elementary states and  $\lambda \cdot \omega$  is the central charge appearing in 4D  $N = 2$  superalgebra. If the central charge vanishes at some point in the moduli space, extra massless BPS states appear there in general. In the heterotic picture this occurs where symmetry enhancement arises through the Frenkel-Kac construction. In the type II picture  $\lambda \cdot \omega$  vanishes if some 2-cycles of the  $K3$  collapse and the  $K3$  develops  $\mathcal{ADE}$  singularities. (*cf.* [21].)

Now we turn to the specific cases of (A), (B) and (C). For these cases the lattice  $\Lambda$  is given respectively by

$$\Lambda_A = L_+ \quad (8)$$

$$\Lambda_B = H' \quad (9)$$

$$\Lambda_C = H' \oplus L_- \quad (10)$$

where  $H'$  is a copy of  $H$  and its basis is denoted by  $\{f_1, f_2\}$  and  $L_\pm$  are the one-dimensional lattices generated by  $\delta_\pm$  with  $\delta_\pm$  satisfying  $\delta_\pm^2 = \pm \frac{1}{2}$ . For cases  $(A')$ ,  $(B')$  and  $(C')$  we should make replacement:  $\Lambda_* \rightarrow \Lambda_* \oplus \mathcal{E}_8(-1)$  where  $\mathcal{E}_8(-1)$  is the negative of an  $\mathcal{E}_8$  root lattice.

Since  $r = 1, 2$ , and  $3$  for  $(A)$ ,  $(B)$  and  $(C)$ , the space  $\mathcal{D}^+$  is given respectively by  $\mathbf{H}_1$ ,  $\mathbf{H}_1 \times \mathbf{H}_1$  and  $\mathbf{H}_2$  where  $\mathbf{H}_1$  is the standard upper half-plane and  $\mathbf{H}_2$  is the Siegel upper half-plane of genus two. The  $T$ -duality groups are respectively  $SL(2, \mathbf{Z})$ ,  $(SL(2, \mathbf{Z}) \times SL(2, \mathbf{Z}))/\mathbf{Z}_2$  and  $Sp(4, \mathbf{Z})$ . We will take the following parametrization of  $y \in \mathcal{D}^+$ :

$$(A) : y = 2T\delta_+, \quad T \in \mathbf{H}_1, \quad (11)$$

$$(B) : y = Tf_1 + Uf_2, \quad (T, U) \in \mathbf{H}_1 \times \mathbf{H}_1, \quad (12)$$

$$(C) : y = Tf_1 + Uf_2 + 2V\delta_-, \quad \Omega = \begin{pmatrix} T & V \\ V & U \end{pmatrix} \in \mathbf{H}_2. \quad (13)$$

To present our formulas of modular forms we have to introduce some notations. First recall that the Jacobi theta functions are defined by

$$\begin{aligned} \vartheta_1(\tau, z) &= i \sum_{n \in \mathbf{Z}} (-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} \zeta^{n-\frac{1}{2}}, & \vartheta_2(\tau, z) &= \sum_{n \in \mathbf{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} \zeta^{n+\frac{1}{2}}, \\ \vartheta_3(\tau, z) &= \sum_{n \in \mathbf{Z}} q^{\frac{n^2}{2}} \zeta^n, & \vartheta_4(\tau, z) &= \sum_{n \in \mathbf{Z}} (-1)^n q^{\frac{n^2}{2}} \zeta^n, \end{aligned} \quad (14)$$

where  $q = e[\tau]$  and  $\zeta = e[z]$ . For later convenience we also introduce the following theta functions:

$$\theta_{\text{ev}}(\tau, z) = \sum_{n \in \mathbf{Z}} q^{\frac{1}{4}(2n)^2} \zeta^{2n}, \quad \theta_{\text{od}}(\tau, z) = \sum_{n \in \mathbf{Z}} q^{\frac{1}{4}(2n+1)^2} \zeta^{2n+1} \quad (15)$$

thus  $\theta_{\text{ev}}(\tau, z) = \vartheta_3(2\tau, 2z)$  and  $\theta_{\text{od}}(\tau, z) = \vartheta_2(2\tau, 2z)$ . We use simplified notations:  $\vartheta_k^0(\tau) = \vartheta_k(\tau, 0)$ ,  $\theta_{\text{ev}}^0(\tau) = \theta_{\text{ev}}(\tau, 0)$  and  $\theta_{\text{od}}^0(\tau) = \theta_{\text{od}}(\tau, 0)$ .

As is well-known the ring of modular forms with respect to  $SL(2, \mathbf{Z})$  is generated by the Eisenstein series of weight four and six:

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \quad E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \quad (16)$$

where  $\sigma_k(n) = \sum_{d|n} d^k$ . In addition, we need the Eisenstein series of “weight two”:

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n = \Theta_q \log \Delta(\tau), \quad (17)$$

where  $\Delta(\tau) = \eta(\tau)^{24}$  and  $\Theta_q$  is the Euler derivative  $q \frac{d}{dq}$ . It satisfies the functional equation  $E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) + \frac{12}{2\pi i} c(c\tau+d)$  for  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$ . These

Eisenstein series are mutually related by

$$\Theta_q E_k = \frac{k}{12} (E_2 E_k - E_{k+2}), \quad (k = 4, 6), \quad (18)$$

where  $E_8 = E_4^2$ . The elliptic modular function is given by

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)} = \frac{E_6(\tau)^2}{\Delta(\tau)} + j(i), \quad j(i) = 1728. \quad (19)$$

For our purpose, it is useful to introduce the following functions (*cf.* [22])

$$\theta(\tau) = \sum_{n \in \mathbf{Z}} q^{n^2/4} = \vartheta_3^0(\tau/2) = \theta_{\text{ev}}^0(\tau) + \theta_{\text{od}}^0(\tau), \quad (20)$$

$$F(\tau) = \sum_{n > 0, n \text{ odd}} \sigma_1(n) q^{n^2/4} = \frac{1}{16} \vartheta_2^0(\tau/2)^4. \quad (21)$$

They satisfy the functional equations

$$\theta(\tau + 4) = \theta(\tau), \quad \theta\left(\frac{\tau}{\tau + 1}\right) = (\tau + 1)^{\frac{1}{2}} \theta(\tau), \quad (22)$$

$$F(\tau + 4) = F(\tau), \quad F\left(\frac{\tau}{\tau + 1}\right) = (\tau + 1)^2 F(\tau), \quad (23)$$

and hence are modular forms with respect to the modular subgroup  $\Gamma^0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) \mid b \equiv 0 \pmod{4} \right\}$ .

In what follows we will give our expressions of modular forms for cases (A), (B) and (C). There are three kinds of (nearly holomorphic) modular forms  $H_*$ ,  $\tilde{H}_*$  and  $J_*$  for  $* = A, B, C$ . The constant terms of  $H_*$ ,  $\tilde{H}_*$  and  $J_*$  are respectively  $\chi$ ,  $\chi - 48$  and 0 where  $\chi$  is the Euler characteristic of the corresponding Calabi-Yau manifold and we remind that  $b_{\text{grav}} = 48 - \chi$  is the gravitational one-loop beta coefficient [23, 24]. The functions  $H_*$ ,  $\tilde{H}_*$  and  $J_*$  are related in such a way that will turn out to be important.

For case (A) our proposed expressions are

$$H_A(\tau) = \frac{2\theta(\tau)E_4(\tau)G_6(\tau)}{\Delta(\tau)} = \sum_{N \in \mathbf{Z} \text{ or } \mathbf{Z} + \frac{1}{4}} c(N) q^N \quad (24)$$

$$= \frac{2}{q} - 252 - 2496 q^{1/4} - 223752 q - 725504 q^{5/4} - \dots \quad (25)$$

$$\tilde{H}_A(\tau) = \frac{2\theta(\tau)E_2(\tau)E_4(\tau)G_6(\tau)}{\Delta(\tau)} = \sum_{N \in \mathbf{Z} \text{ or } \mathbf{Z} + \frac{1}{4}} \tilde{c}(N) q^N \quad (26)$$

$$= \frac{2}{q} - 300 - 2496 q^{1/4} - 217848 q - 665600 q^{5/4} - \dots \quad (27)$$

$$J_A(\tau) = 2\theta(\tau) \left( \frac{E_6(\tau)G_6(\tau)}{\Delta(\tau)} + 870 \right) = \sum_{N \in \mathbf{Z} \text{ or } \mathbf{Z} + \frac{1}{4}} a(N)q^N \quad (28)$$

$$= \frac{2}{q} + 984q^{1/4} + 286752q + 1131520q^{5/4} + \dots \quad (29)$$

where

$$G_6(\tau) = E_6(\tau) - 2F(\tau)(\theta(\tau)^4 - 2F(\tau))(\theta(\tau)^4 - 16F(\tau)), \quad (30)$$

and we have the relation

$$-\frac{24}{3}\Theta_q H_A(\tau) = \tilde{H}_A(\tau) + 7J_A(\tau) + 300\theta(\tau). \quad (31)$$

Similarly for  $(B)$  we have [7]

$$H_B(\tau) = \frac{2E_4(\tau)E_6(\tau)}{\Delta(\tau)} = \sum_{N \in \mathbf{Z}} c(N)q^N \quad (32)$$

$$= \frac{2}{q} - 480 - 282888q - 17058560q^2 - \dots \quad (33)$$

$$\tilde{H}_B(\tau) = \frac{2E_2(\tau)E_4(\tau)E_6(\tau)}{\Delta(\tau)} = \sum_{N \in \mathbf{Z}} \tilde{c}(N)q^N \quad (34)$$

$$= \frac{2}{q} - 528 - 271512q - 10234880q^2 - \dots \quad (35)$$

$$J_B(\tau) = 2 \left( \frac{E_6(\tau)^2}{\Delta(\tau)} + 984 \right) = \sum_{N \in \mathbf{Z}} a(N)q^N \quad (36)$$

$$= \frac{2}{q} + 393768q + 42987520q^2 + \dots \quad (37)$$

and

$$-\frac{24}{4}\Theta_q H_B(\tau) = \tilde{H}_B(\tau) + 5J_B(\tau) + 528. \quad (38)$$

To present our expressions for  $(C)$ , some familiarity with Jacobi forms [25] is needed. A Jacobi form  $\Phi_{k,m}$  of weight  $k$  and index  $m$  satisfies

$$\Phi_{k,m} \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^k \mathbf{e} \left[ \frac{mcz^2}{c\tau + d} \right] \Phi_{k,m}(\tau, z), \quad (39)$$

$$\Phi_{k,m}(\tau, z + \lambda\tau + \mu) = \mathbf{e} \left[ -m(\lambda^2\tau + 2\lambda z) \right] \Phi_{k,m}(\tau, z), \quad (40)$$

where  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$  and  $\lambda, \mu \in \mathbf{Z}$ . The ring of Jacobi forms of index 1 is generated by the Jacobi-Eisenstein series  $E_{4,1}$  of weight 4 and  $E_{6,1}$  of weight 6 which have expansions

$$E_{4,1}(\tau, z) = 1 + \left( \frac{1}{\zeta^2} + \frac{56}{\zeta} + 126 + 56\zeta + \zeta^2 \right) q + \dots \quad (41)$$



$$E_{6,1}(\tau, z) = 1 + \left( \frac{1}{\zeta^2} - \frac{88}{\zeta} - 330 - 88\zeta + \zeta^2 \right) q + \dots \quad (42)$$

The  $K3$  elliptic genus  $Z(\tau, z)$  is a (weak) cusp Jacobi form of weight 0 and index 1 given by

$$\begin{aligned} Z(\tau, z) &= \frac{1}{72} \frac{E_4(\tau)^2 E_{4,1}(\tau, z) - E_6(\tau) E_{6,1}(\tau, z)}{\Delta(\tau)} \\ &= \frac{2}{\zeta} + 20 + 2\zeta + \left( \frac{20}{\zeta^2} - \frac{128}{\zeta} + 216 - 128\zeta + 20\zeta^2 \right) q + \dots \end{aligned} \quad (43)$$

A general theory shows that any Jacobi form  $\Phi_{k,1}$  of index 1 can be decomposed as

$$\Phi_{k,1}(\tau, z) = \Phi_{k,1}^{\text{ev}}(\tau) \theta_{\text{ev}}(\tau, z) + \Phi_{k,1}^{\text{od}}(\tau) \theta_{\text{od}}(\tau, z), \quad (44)$$

and we introduce

$$\hat{\Phi}_{k,1}(\tau) := \Phi_{k,1}^{\text{ev}}(\tau) + \Phi_{k,1}^{\text{od}}(\tau), \quad (45)$$

for such a decomposition. Explicitly we have [26]

$$Z(\tau, z) = Z^{\text{ev}}(\tau) \theta_{\text{ev}}(\tau, z) + Z^{\text{od}}(\tau) \theta_{\text{od}}(\tau, z) \quad (46)$$

$$\begin{aligned} Z^{\text{ev}}(\tau) &= \frac{6\{\vartheta_2^0(\tau) \vartheta_4^0(\tau)\}^2 \theta_{\text{ev}}^0(\tau) - 2(\vartheta_4^0(\tau)^4 - \vartheta_2^0(\tau)^4) \theta_{\text{od}}^0(\tau)}{\eta(\tau)^6} \\ &= 20 + 216q + 1616q^2 + 8032q^3 + \dots \end{aligned} \quad (47)$$

$$\begin{aligned} Z^{\text{od}}(\tau) &= \frac{6\{\vartheta_2^0(\tau) \vartheta_4^0(\tau)\}^2 \theta_{\text{od}}^0(\tau) + 2(\vartheta_4^0(\tau)^4 - \vartheta_2^0(\tau)^4) \theta_{\text{ev}}^0(\tau)}{\eta(\tau)^6} \\ &= \frac{2}{q^{1/4}} - 128q^{3/4} - 1026q^{7/4} - 5504q^{11/4} - \dots, \end{aligned} \quad (48)$$

and

$$E_{4,1}(\tau, z) = E_{4,1}^{\text{ev}}(\tau) \theta_{\text{ev}}(\tau, z) + E_{4,1}^{\text{od}}(\tau) \theta_{\text{od}}(\tau, z) \quad (49)$$

$$\begin{aligned} E_{4,1}^{\text{ev}}(\tau) &= \theta_{\text{ev}}^0(\tau)^7 + 7\theta_{\text{ev}}^0(\tau)^3 \theta_{\text{od}}^0(\tau)^4 \\ &= 1 + 126q + 756q^2 + \dots \end{aligned} \quad (50)$$

$$\begin{aligned} E_{4,1}^{\text{od}}(\tau) &= \theta_{\text{od}}^0(\tau)^7 + 7\theta_{\text{od}}^0(\tau)^3 \theta_{\text{ev}}^0(\tau)^4 \\ &= 56q^{3/4} + 576q^{7/4} + 1512q^{11/4} + \dots \end{aligned} \quad (51)$$

$$E_{6,1}(\tau, z) = E_{6,1}^{\text{ev}}(\tau) \theta_{\text{ev}}(\tau, z) + E_{6,1}^{\text{od}}(\tau) \theta_{\text{od}}(\tau, z) \quad (52)$$

$$\begin{aligned} E_{6,1}^{\text{ev}}(\tau) &= -\frac{1}{4} \left[ \vartheta_2^0(\tau)^6 Z^{\text{ev}}(\tau) - (\vartheta_3^0(\tau)^6 + \vartheta_4^0(\tau)^6) Z^{\text{od}}(\tau) \right] \eta(\tau)^6 \\ &= 1 - 330q - 7524q^2 - \dots \end{aligned} \quad (53)$$

$$\begin{aligned} E_{6,1}^{\text{od}}(\tau) &= -\frac{1}{4} \left[ -\vartheta_2^0(\tau)^6 Z^{\text{od}}(\tau) + (\vartheta_3^0(\tau)^6 - \vartheta_4^0(\tau)^6) Z^{\text{ev}}(\tau) \right] \eta(\tau)^6 \\ &= -88q^{3/4} - 4224q^{7/4} - 30600q^{11/4} - \dots \end{aligned} \quad (54)$$

With this preparation we can write down the expressions for (C):

$$H_C(\tau) = \frac{2E_4(\tau)\hat{E}_{6,1}(\tau)}{\Delta(\tau)} = \sum_{N \in \mathbf{Z} \text{ or } \mathbf{Z} + \frac{3}{4}} c(N)q^N \quad (55)$$

$$= \frac{2}{q} - \frac{176}{q^{1/4}} - 132 - 54912 q^{3/4} - 172800 q - 3742416 q^{7/4} - \dots \quad (56)$$

$$\tilde{H}_C(\tau) = \frac{2E_2(\tau)E_4(\tau)\hat{E}_{6,1}(\tau)}{\Delta(\tau)} = \sum_{N \in \mathbf{Z} \text{ or } \mathbf{Z} + \frac{3}{4}} \tilde{c}(N)q^N \quad (57)$$

$$= \frac{2}{q} - \frac{176}{q^{1/4}} - 180 - 50688 q^{3/4} - 169776 q - 2411856 q^{7/4} - \dots \quad (58)$$

$$J_C(\tau) = \frac{2E_6(\tau)\hat{E}_{6,1}(\tau)}{\Delta(\tau)} + 81\hat{Z}(\tau) = \sum_{N \in \mathbf{Z} \text{ or } \mathbf{Z} + \frac{3}{4}} a(N)q^N \quad (59)$$

$$= \frac{2}{q} - \frac{14}{q^{1/4}} + 65664 q^{3/4} + 262440 q + 8909838 q^{7/4} + \dots \quad (60)$$

Again there exists a relation among these functions, *i.e.*

$$- \frac{24}{5} \Theta_q H_C(\tau) = \tilde{H}_C(\tau) + \frac{19}{5} J_C(\tau) + 9\hat{Z}(\tau). \quad (61)$$

This follows from

$$(\Theta_q - \frac{1}{4}\Theta_\zeta^2) E_{k,1} = \frac{2k-1}{24} (E_2 E_{k,1} - E_{k+2,1}), \quad (k=4,6), \quad (62)$$

where  $E_{8,1} = E_4 E_{4,1}$  and

$$(\Theta_q - \frac{1}{4}\Theta_\zeta^2) \theta_{\text{ev}} = (\Theta_q - \frac{1}{4}\Theta_\zeta^2) \theta_{\text{od}} = 0. \quad (63)$$

Having presented our expressions for modular forms, we can now discuss the physical implications of the coefficients of these modular forms. The heterotic prepotential  $^{\text{het}}\mathcal{F}$  assumes the form

$$^{\text{het}}\mathcal{F}(S, y) = \frac{1}{2} S y^2 + v(y) + \mathcal{F}_{NP}(\mathbf{e}[S], y). \quad (64)$$

As in [7], the perturbative contribution  $v(y)$  should be written in terms of the coefficients of  $H_*$  as

$$v(y) = p(y) - \frac{1}{(2\pi i)^3} \sum_{\alpha \in \Lambda, \alpha > 0} c(\alpha^2/2) \text{Li}_3(\mathbf{e}[\alpha \cdot y]), \quad (65)$$

where  $\alpha > 0$  means that

$$\begin{aligned} (A) : & \quad n > 0, \\ (B) : & \quad (i) \ k > 0, \text{ or } (ii) \ k = 0, \ l > 0, \\ (C) : & \quad (i) \ k > 0, \text{ or } (ii) \ k = 0, \ l > 0, \text{ or } (iii) \ k = l = 0, \ b < 0, \end{aligned} \quad (66)$$

if  $\alpha$  is parametrized as  $(A) : \alpha = n\delta_+$ ,  $(B) : \alpha = lf_1 + kf_2$ ,  $(C) : \alpha = lf_1 + kf_2 - b\delta_-$ . The term  $p(y)$  is a chamber-dependent [7] cubic polynomial and for each case we can take<sup>||</sup>

$$p_A(y) = \frac{2}{3}T^3 - T^2 - \frac{13}{6}T, \quad (67)$$

$$p_B(y) = \frac{1}{3}U^3 - \frac{11}{6}U^2T - UT^2, \quad (68)$$

$$p_C(y) = p_B(y) - \frac{31}{6}UV^2 - 5TV^2 + \frac{43}{6}TUV + \frac{37}{6}V^3, \quad (69)$$

where  $p_B$  and  $p_C$  are evaluated in a chamber where  $\text{Im}T > \text{Im}U$ . However  $p(y)$  is ambiguous due to the freedom of adding quadratic polynomials in the components of  $\omega(y)$ . Thus, for instance, we have

$$p_C(y) \sim \frac{1}{3}U^3 - 6TV^2 - 7UV^2 + \frac{40}{3}V^3. \quad (70)$$

Next we turn to  ${}^{\text{het}}\mathcal{F}_1$  which is the heterotic equivalent of  ${}^{\text{II}}F_1^{\text{top}}$ . For this purpose we need several product formulas of automorphic forms on  $T$ -duality groups. Such product representations have recently been the subject of intensive study by several mathematicians [22, 27–29] and possible connections with the denominator functions of generalized Kac-Moody (super) algebras have been discussed. For a given series  $\varphi(\tau) = \sum_N c(N)q^N$  introduce an infinite product  $\Psi$  by

$$\Psi[\Lambda, \rho, \varphi] = \mathbf{e}[\rho \cdot y] \prod_{\alpha \in \Lambda, \alpha > 0} (1 - \mathbf{e}[\alpha \cdot y])^{c(\alpha^2/2)}, \quad (71)$$

then what is relevant to us may be summarized as:

$\Lambda$	$\varphi$	$\rho$	$\Psi$
$\Lambda_A$	$J_A(\tau)$	$-\delta_+$	$j(T) - j(i)$
$\Lambda_A$	$\theta(\tau)$	$\frac{1}{12}\delta_+$	$\eta(T)^2$
$\Lambda_B$	$J_B(\tau)$	$-2f_2$	$(j(T) - j(U))^2$
$\Lambda_B$	1	$\frac{1}{24}(f_1 + f_2)$	$\eta(T)\eta(U)$
$\Lambda_C$	$J_C(\tau)$	$-f_1 - 3f_2 + 5\delta_-$	$\frac{\Delta_{35}(\Omega)^2}{\Delta_5(\Omega)^{14}}$
$\Lambda_C$	$\hat{Z}(\tau)$	$f_1 + f_2 - \delta_-$	$\Delta_5(\Omega)^2$

(72)

where we assumed  $\text{Im}T > \text{Im}U$ . The functions  $\Delta_{35}(\Omega)$  and  $\Delta_5(\Omega)$  are related to the Igusa cusp forms [30],  $\chi_{35}(\Omega)$  and  $\chi_{10}(\Omega)$  by the relations  $\Delta_{35}(\Omega) = 4i\chi_{35}(\Omega)$  and  $\Delta_5(\Omega)^2 =$

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<sup>||</sup> $p_B(y)$  was calculated in [7]. A similar calculation leads to  $p_C(y)$ .

$-4\chi_{10}(\Omega)$ . The first and third results in (72) are due to Borchers [22], while the last one is due to Gritsenko and Nikulin [27, 28]. The Jacobi form

$$\begin{aligned} X(\tau, z) &= \frac{E_6(\tau)E_{6,1}(\tau, z)}{\Delta(\tau)} + 44Z(\tau, z) \\ &= \frac{1}{q} + \left( \frac{1}{\zeta^2} + 70 + \zeta^2 \right) \\ &\quad + \left( \frac{70}{\zeta^2} + \frac{32384}{\zeta} + 131976 + 32384\zeta + 70\zeta^2 \right) q + \dots \end{aligned} \quad (73)$$

coincides with the last equation in ref. [29], namely,  $\phi_{0,1}|_0 T_0(2) - 2\phi_{0,1}$  in the notation there. Consequently, the fifth result also follows from their result. The correspondence  $Z(\tau, z) \leftrightarrow \Delta_5(\Omega)^2$  can be confirmed by a calculation of threshold correction [31]. Similarly I have checked the correspondence  $X(\tau, z) \leftrightarrow \Delta_{35}(\Omega)$  by an evaluation of the pertinent modular integral following the approach of [32] [7].

If we separate the heterotic free energy  $^{\text{het}}\mathcal{F}_1$  into the perturbative and non-perturbative parts as

$$^{\text{het}}\mathcal{F}_1(S, y) = -\frac{2\pi i}{12} f^W(S, y) + \mathcal{F}_1^{\text{NP}}(\mathbf{e}[S], y), \quad (74)$$

then one may infer that

$$f_A^W(S, y) = 24\tilde{S} + \frac{2}{2\pi i} \left[ 7 \log(j(T) - j(i)) + 300 \log \eta(T)^2 \right], \quad (75)$$

$$f_B^W(S, y) = 24\tilde{S} + \frac{2}{2\pi i} \left[ 5 \log(j(T) - j(U))^2 + 528 \log(\eta(T)\eta(U)) \right], \quad (76)$$

$$f_C^W(S, y) = 24\tilde{S} + \frac{2}{2\pi i} \left[ \frac{19}{5} \log(\Delta_{35}(\Omega)^2 / \Delta_5(\Omega)^{14}) + 9 \log \Delta_5(\Omega)^2 \right]. \quad (77)$$

In these equations,

$$\tilde{S} = S + \frac{1}{r+2} \nabla_y^2 v(y), \quad (78)$$

is the invariant dilaton [17] [7], where  $\nabla_y^2$  is a second-order differential operator satisfying  $\nabla_y^2 \mathbf{e}[\alpha \cdot y] = (2\pi i)^2 \alpha^2 \mathbf{e}[\alpha \cdot y]$  and is explicitly given by (A) :  $\nabla_y^2 = \frac{1}{2} \partial_T^2$ , (B) :  $\nabla_y^2 = 2\partial_T \partial_U$  and (C) :  $\nabla_y^2 = 2(\partial_T \partial_U - \frac{1}{4} \partial_V^2)$ . One can easily see that the expressions (75)–(77) for  $f^W$  have physically acceptable modular properties with respect to  $T$ -duality as Wilsonian gravitational couplings. Actually the expressions for  $f_A^W$  and  $f_B^W$  can be seen to agree with the results in [2, 3]. Using the relations (31), (38), (61) and the product formulas (72) one can deduce for all the cases we are considering that

$$f^W(S, y) = 24S + \ell(y) + \frac{2}{2\pi i} \sum_{\alpha \in \Lambda, \alpha > 0} \tilde{c}(\alpha^2/2) \text{Li}_1(\mathbf{e}[\alpha \cdot y]), \quad (79)$$

where  $\ell(y)$  is linear in  $y$  and has an ambiguity due to that of  $p(y)$ .

In order to test the duality conjectures we must compare (64) and (65) against (1) as well as (74) and (79) against (2) by judiciously identifying linear combinations of  $t^i$ 's with  $S$  and  $y$ . For case (B) this has already been done in [7, 10]. For  $(A)^{**}$ , the comparison of (74) and (79) with (2) leads to<sup>††</sup>

$$N^{r,e}(n, 0) = N^r(n, 0) + 12 \sum_{d|n} N^e(d, 0) = -\tilde{c}(n^2/4), \quad (n \geq 1), \quad (80)$$

where our choice of the identification rule is such that  $t^1 = T$  and  $t^2 = S$ . Thus we obtain conjectured relations for (A):

$$N^r(n, 0) = -c(n^2/4), \quad (81)$$

$$N^e(n, 0) = \frac{1}{12} \sum_{d|n} \mu\left(\frac{n}{d}\right) [c(d^2/4) - \tilde{c}(d^2/4)], \quad (82)$$

where  $n \geq 1$  and  $\mu(\cdot)$  is the classical Möbius function. These give

$n$	$N^r(n, 0)$	$N^e(n, 0)$
1	2496	0
2	223752	-492
3	38637504	-1465984
4	9100224984	-1042943028
5	2557481027520	-595277880960
6	805628041231176	-316194811079664
7	274856132550917568	-163214406650542848
8	99463554195314072664	-83229690442895106144

in perfect agreement with the type IIA results obtained in [33–35].

As for case (C), before attempting any comparison toward establishment of the conjecture, we need results from type IIA calculations for this case, which, as stated earlier, are yet to be done. But we wish to remark one more point about the case (C). Our proposed formulas of modular forms are related to the calculation of threshold corrections on the heterotic side. Analogously to the case (B) treated in [7], one may consider the threshold corrections to the gauge couplings for  $\mathcal{E}_8$  and  $SO(12)$ . The one-loop beta function coefficients for these gauge groups must appear as the constant terms of

$$\begin{aligned} & -\frac{1}{12} \left( \frac{E_2(\tau)E_4(\tau)\hat{E}_{6,1}(\tau) - E_6(\tau)\hat{E}_{6,1}(\tau)}{\Delta(\tau)} \right) \\ & = -60 + 5280 q^{3/4} + 17280 q + \dots, \end{aligned} \quad (83)$$

and

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<sup>\*\*</sup>We should emphasize that the first quantitative test of duality conjecture for (A) [1] was done in [2].

<sup>††</sup> For notational simplicity we write  $N^*(n, m)$  instead of  $N^*((n, m))$  for  $(n, m) \in \mathbf{S}$ .

$$\begin{aligned}
& -\frac{1}{12} \left( \frac{E_2(\tau)E_4(\tau)\hat{E}_{6,1}(\tau) - E_4(\tau)^2\hat{E}_{4,1}(\tau)}{\Delta(\tau)} \right) \\
& = 12 \frac{1}{q^{1/4}} + 60 + 4512 q^{3/4} + \dots .
\end{aligned} \tag{84}$$

This is actually the case since  $b_{\mathcal{E}_8} = -\mathcal{I}(\mathbf{248}) = -60$  and  $b_{SO(12)} = 10\mathcal{I}(\mathbf{32}_s) - \mathcal{I}(\mathbf{66}) = 10 \cdot 8 - 20 = 60$  where  $\mathcal{I}(\mathbf{rep})$  denotes the index of an irrep  $\mathbf{rep}$ .

Though we have restricted ourselves to specific cases in this work, it may well be true that formulas such as (65) and (79) have universal meanings in perturbation theory of heterotic strings on  $K3 \times T^2$  as already advocated for the case of the prepotential in [7]. Also there seems to be still much room for clarifying possible roles of generalized Kac-Moody (super) algebras in a more profound understanding of string duality.

Finally we comment on the cases  $(A')$ ,  $(B')$  and  $(C')$ . The modular forms  $H_{A'}$ ,  $H_{B'}$  and  $H_{C'}$  are obtained by dropping  $E_4$  from  $H_A$ ,  $H_B$  and  $H_C$  since  $E_4$  is the theta series for the  $\mathcal{E}_8$  root lattice<sup>‡‡</sup>:

$$H_{A'}(\tau, z) = \frac{2\theta(\tau)G_6(\tau)}{\Delta(\tau)} \tag{85}$$

$$= \frac{2}{q} - 732 - 2496 q^{1/4} - 52392 q - 126464 q^{5/4} - \dots \tag{86}$$

$$H_{B'}(\tau, z) = \frac{2E_6(\tau)}{\Delta(\tau)} \tag{87}$$

$$= \frac{2}{q} - 960 - 56808 q - 1364480 q^2 - \dots \tag{88}$$

$$H_{C'}(\tau, z) = \frac{2\hat{E}_6(\tau)}{\Delta(\tau)} \tag{89}$$

$$= \frac{2}{q} - \frac{176}{q^{1/4}} - 612 - 12672 q^{3/4} - 30240 q - 320976 q^{7/4} - \dots \tag{90}$$

The constant terms correctly reproduce the Euler characteristics of the corresponding Calabi-Yau manifolds.

*Note added.* While finishing this paper, two related papers [36, 37] appeared on the hep-th archive.

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<sup>‡‡</sup> The case  $(B')$  was already given in [7].

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